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# RATIONAL TOPOLOGY OF CANTILEVER STEEL BEAMS WITH VARIABLE FLANGE WIDTH AND WEB HEIGHT UNDER DEFLECTION AND STRENGTH CONSTRAINTS

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**Abstract.** The article solves the problem of selecting the optimal topology of a cantilever steel I-beam with variable web height and flange width, subject to deflection constraints and assuming an optimal distribution of steel in each cross-section based on strength conditions. The problem is solved using the method of Lagrange multipliers. The optimal design criterion is taken to be the objective function minimizing steel consumption for the structure. The condition of optimal distribution of steel between the flanges and the web based on strength criteria is adopted for each cross-section. Possible deviations from the optimal ratio between the flange area and the web area are accounted for by an additional coefficient. The problem belongs to nonlinear programming. The strength condition of the web is considered inactive and is ensured by structural measures (web stiffeners). Out-of-plane stability of the beam (against lateral bending) is provided by an appropriate system of horizontal bracing along the flanges.

An analytical function describing the cross-sectional variation along the length of a cantilever tapered I-beam is obtained under the optimization conditions for a uniformly distributed load and a given relative design deflection. The derived analytical function for the relative optimal height of the I-beam along the length of the structure is a power-law function and depends on the load, the deflection constraints, and the optimal or rational distribution of steel in each cross-section. The optimal height of the I-beam for the support section (where the maximum bending moment occurs) is determined under the conditions of the problem.

The identified patterns of variation in the optimal beam height allow one to select the optimal topology of the structure and to account for the possibility of higher stresses arising in sections of smaller height. It is confirmed that the optimal structural solution depends on the load distribution law. The obtained results allow determining the degree of variability of the cross-section height for the optimal topology.

The analytical formulas derived for the optimal height of a beam with variable flange width and variable web height enable, at the first stage of variant design, an evaluation of the efficiency of the design solution.

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## PROBLEM STATEMENT

The use of steel structures in the construction of industrial and civil facilities underscores the technological level of countries. The economic feasibility of using steel structures has been proven for long spans under substantial loads when deflection constraints are taken into account.

Moreover, the recyclability of steel results in an integrated reduction of total $CO_2$ emissions over the service life of a building, which is of significant environmental importance.

In the course of research into selecting the optimal topology of an I-beam with a variable web and flange cross-section along its length, a number of constraints arise that must be addressed in combination. First, in beams of variable cross-section, the maximum stresses do not always occur in the section where the maximum bending moment acts [7,8,18,27]. Second, when solving problems considering limitations on allowable deflections, it is necessary to account for the variation of stresses along the beam's length. For steel I-beams of constant cross-section, deflection constraints do not depend on the steel strength or the stress distribution along the length of the structure. However, for beams with variable cross-section, one must consider that determining the minimum steel consumption is related to both the stresses and the deflections of the structure. Therefore, solving such a problem requires additional information for design. A traditional approach in optimal beam design by steel consumption allows the optimal web height of an I-beam to be determined using analytical formulas derived in the process of minimizing steel usage. This approach is based on the idea that for a given design bending moment in a certain cross-section, increasing the web height increases the section modulus, but the steel usage for the web increases as well.

Conversely, increasing the web height reduces the steel consumption in the flanges of the I-beam. However, both of these trends are nonlinear. The contribution of the flanges to the section modulus is greater than that of the web; yet as the web height increases, the required flange area decreases, and the flanges' contribution to further increasing the section modulus (due to the greater distance from the neutral axis) diminishes. Therefore, there exists a web height at which the steel consumption for the profile is minimal. With a varying flange width and web height, the search for a rational (constrained optimal) web height is formulated as a nonlinear programming problem aimed at minimizing steel consumption with deflection constraints. Solving such an optimal design problem is important for conceptual design considering the requirements of two limit states. Since the extremum of such problems is rather flat, it is assumed that suitable plate thicknesses for fabricating the beam are always available from standard stock.

There are numerous studies related to determining the cross-sections of I-beams with variable web height. For example, beams have been investigated under strength and stability conditions under axial compression, including potential loss of stability in the plane of bending and the occurrence of bimoment stresses [1,2,4,7,9,12,16,27,28].

However, research into the optimal design of I-beams with variable web height is developing further due to the widespread use of such members in construction with steel frameworks employing tapered members [7,9,18,26,27,29].

The theory of optimal design and various scientific approaches to finding optimal structural solutions for steel structures are presented in works [3,18,23,28,29]. An important factor in choosing the best structural topology is the choice of the optimal design criterion. The most relevant criterion is minimizing steel consumption (material usage) [2, 5, 7, 8, 14, 15,18, 21, 22, 23, 27, 29] An optimality criterion based on energy principles is substantiated in work [19]. Optimization by cost and environmental impact has been studied in [19,20,23,28]. The criterion of rational design for combined roof trusses—based on equalizing the bending moment-induced stresses in the top chord of an I-section, taking into account an energy criterion—is given in [15,23]. The developed optimal design criteria also extend to other efficient steel beam structures: beams with corrugated webs [13,16, 25] and beams with perforated webs [13,16,30].

Methods used to select the optimal structural topology are generally divided into analytical approaches (with numerical investigations of the obtained criteria and algorithmic constraint functions) [5,7,8,15,16,23,30,31,32]. This approach is used when it is possible to describe the continuity of the discrete range of steel plate stock and the interrelation between different parameters of cross-section variability and topology along the beam's length.

The use of genetic algorithms is widespread and is among the most common approaches [16,21,22,23,25, 30,31,32], However, finding a global minimum requires additional research. Furthermore, if there is a possibility of multiple optimal solutions (differences on the order of 5–6% in steel consumption depending on the discretization of available plate thicknesses), it becomes necessary to further develop and adapt existing optimization methods Since switching to a different steel section (product range) can lead to significant changes in the structural topology, the finite element method—using various algorithms—is typically applied at the detailed design stage and yields reliable results, although it is computationally intensive. This approach is generally used during the working design phase of steel structures. However, reliable results can be obtained more rapidly if the initial dimensions of the support cross-section are rational (close to optimal). Such input parameters can be derived from generalized analytical studies in optimal design conducted during the conceptual or preliminary variant-based design stage of steel structures.

The relevance of developing optimal design solutions for I-beams with variable cross-section is also driven by the potential to reduce costs under thermal loading during fire conditions, as such optimization allows for lower expenses on fire protection for critical structural elements. [6,11,24].

In its general formulation, the task of finding the optimal topology of a steel beam with variable web height and variable flange width is a multi-criteria optimization problem that requires further development and adaptation of existing optimization methods [14,23,28,30]. The relevance of such problems increases particularly under the influence of dynamic loading [5,17] and the activation of displacement constraints [7,8,23,28,29,32]. Thus, the selection of an optimal topology for a welded I-beam with variable flange width and web height is a pressing and significant research direction. The study of flexible structural elements [10] should be considered as a separate but related problem.

The relevance of the stated problem and the conducted analytical studies [7,8] may also be applied to evaluate the rationality of the structural solution under the development of localized plastic deformations along the cross-sectional height.

The literature review has shown that insufficient analytical research has been conducted in this area, despite its importance at the initial stage of conceptual (variant-based) design.

The relevance of the following research lies in its ability to reveal the influence of stiffness variation patterns on the stress–strain state of the structure and to open new approaches for solving subsequent rational design problems..

## MAIN STUDY

The study addresses the problem of selecting the optimal topology of an elastic steel I-beam with span length $(l)$ featuring variable flange width ($b_{f,z}$, $b_{f,z} = b_{f,0}(1 \mp \gamma_b z / l)^s$) and variable web height ($h_z$,). The origin of the Cartesian coordinate system is placed at the center of the largest cross-section. The web slenderness varies along the beam's length and is defined as $\lambda_w = h_0 t_w$ The analysis of the beam's bending behavior is based on the Euler–Bernoulli bending hypotheses [2,7,8].

The variation of the flange width and the web height of the I-beam is assumed to follow a power-law distribution with defined cross-sectional gradients.

$$b_{f,z} = b_{f0}(1 \mp \gamma_{fb}\frac{z}{l})^s \rightarrow \frac{z}{l} = 1 \rightarrow b_{f,n} = b_{f0}(1 \mp \gamma_{fb})^s$$

$$\gamma_{fb} = \mp \sqrt[s]{1 - \frac{b_{fn}}{b_{f0}}}.$$

$$h_z = h_0(1 - \gamma_h \frac{z}{l})^s \rightarrow \frac{z}{l} = 1 \rightarrow h_n = h_0(1 - \gamma_h)^s;$$

$$\gamma_h = \sqrt[s]{\frac{h_0}{h_n}}.$$

To simplify notation, generalized functions describing the variation of the cross-section are introduced. The cross-sectional area of the steel

I-beam is considered to depend on the variation of the web height and flange width.

$$h_z = h_0 f_{h,z};\quad f_{h,z} = (1-\gamma_h \frac{z}{l})^s.$$

$$b_{f,z} = b_{f0} f_{b,z};\quad f_{b,z} = (1\mp\gamma_{fb} \frac{z}{l})^s. \qquad (1)$$

$$2A_{f,z} = 2b_{f,z}t_f;\qquad A_{w,z} = h_z t_w.$$

$$A_z = 2A_{f,z} + A_{w,z}.\qquad A_z = 2b_{f,z}t_f + h$$

In the formulas provided, the following notations are used: $t_w$ – thickness of the I-beam web, $t_f$ – thickness of the I-beam flange, $l$ – length of the beam, $\gamma_{fb}$ $\gamma_{bx}$ – gradient (parameter) of flange width variation, $b_{f,0}$ – flange width at z=0 (maximum width), $b_{f,n}$ – flange width at z=l (minimum width), через $t_z = z/l$ - dimensionless longitudinal coordinate of the cross-section, with the origin located at the section of maximum size.

The geometric properties of the symmetric cross-section of the steel I-beam under variable flange width are denoted as follows: $I_{xz}; I_{x0}; I_{xn}$ - current moment of inertia of the cross-section, initial (maximum) moment of inertia, minimum moment of inertia along the beam length

$$I_{xz} = 2b_{f0}(1-\gamma_{fb}t_z)t_f \frac{h_0^2}{4} + \frac{h_0^3 t_w}{12}.$$

$$I_{x0} = 2b_0 t_f \frac{h_0^2}{4} + \frac{h_0^3 t_w}{12} = \frac{h_0^2}{2} b_0 t_f \left(1 + \frac{h_0 t_w}{6b_0 t_f}\right).$$

Accordingly, the variation of the moments of inertia of the steel beam cross-sections with changing flange width, relative to the main centroidal axis Ox, can be expressed as follows:

$$I_{x,z} = \left(\frac{A_{f,0}h_z^2}{2} + \frac{t_w h_z^3}{12}\right) = \frac{t_w h_z^3}{2}\left(\frac{A_{f,z}}{t_w h_z} + \frac{1}{6}\right).$$

It is important to assume a constant web thickness, which is determined based on the slenderness of the maximum cross-section.

$$t_w = \frac{h_0}{\lambda_w}$$

In the generalized form, the moment of inertia of the cross-section has the following analytical expression.

$$I_{x,z} = \left(h_0 f_{hz}\right)^3 \frac{h_0}{2\lambda_w}\left(\frac{t_f b_{f,0}\lambda_w}{h_0^2}\frac{f_{bz}}{f_{hz}} + \frac{1}{6}\right) \qquad (2)$$

The deflection constraint condition for the structure is defined using Mohr's integral, which takes the following form.

$$u_\eta\left(h_z, t_{w,z}, A_{f,z}\right) = .\int_0^l \frac{M_{x,z}M_{x,\mathrm{P},z}}{EI_{x,z}}dz \le \Delta_\eta.$$

$$M_{x,P,z} = P_1 l\left(1 - z/l\right);\quad M_{x,z} = \frac{ql^2}{2} f_{M,z}^m.$$

$$f_{M,z}^m = \left(1 - \frac{z^n}{l^n}\right)^m . f_{M,z}^m\left(1 - z/l\right) \qquad (3)$$

$$\frac{ql^4}{2}.\int_0^1 \frac{f_{M,z}^m\left(1 - z/l\right)}{EI_{x,z}} d\frac{z}{l} \le \Delta_\eta$$

The methodology for determining the minimum steel consumption under strength and deflection constraints is developed for a tapered cantilever I-beam with variable web height and flange width.

The steel consumption function accounts for the inclusion of the web thickness parameter through its dependence on the maximum height $h_0$ and the web slenderness $\lambda_w$.

The optimal design problem for steel beams with variable cross-section (4) is formulated as a nonlinear mathematical programming problem [2, 3, 7, 8, 23].

$$m_{b,z} = 2\rho l b_{f,0} t_f \int_l f_{bz}\, dz + \rho l h_{w,0} t_w \int_l f_{hz}\, dz \to min \qquad (4)$$

The objective function for minimizing the steel mass (4) incorporates all geometric parameters of the beam's cross-section and their variation along the beam's length (2). The deflection constraints are adopted as the governing restrictions, where $\Delta_\eta$ denotes the limiting displacements of the structure. Displacements are determined using Mohr's integral (3).

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The condition that the structure's deflections do not exceed the allowable limits depends on the variability of the geometric properties of the I-section. Therefore, in its general form, this condition can be expressed as a function of the optimal height determined based on minimum steel consumption under strength conditions. Applying the optimality condition for I-beams [7,8,16,23] if, in each current cross-section, the total area of the two flanges equals the area of the web, this constitutes a sufficient condition for an optimal cross-section. If every cross-section along the length of the structure is optimal, then the entire structure is optimal in terms of steel usage

$$2A_{f,z} = t_w h_z = \frac{h_0^2 f_{h,z}}{\lambda_w} = \frac{h_0 h_z}{\lambda_w}.$$

However, due to the discreteness of the available rolled steel sections (in terms of thickness) and the need to satisfy the web stability conditions, a correction factor ($k_b$) is introduced for the flange area. Designers are often forced to deviate from the optimal web height by adjusting the flange width and area. Therefore, in subsequent research, it is assumed that the rational distribution of steel along the cross-sectional height of the I-beam between the web and the flanges will follow the condition below.

$$\frac{2t_f b_{f,0}\lambda_w}{(h_0 f_{hz})h_0} = k_b \rightarrow \frac{t_f b_{f,0}\lambda_w}{(h_0 f_{hz})h_0} = \frac{1}{2}k_b. \quad (5)$$

Thus, taking into account the rationality condition for each cross-section of the beam, the moment of inertia (2) takes the following form.

$$I_{x,z} = (h_0 f_{hz})^3 \frac{h_0}{2\lambda_w}\left(\frac{1}{2}k_b + \frac{1}{6}\right) \quad (6)$$

The generalized expression for deflections is obtained using Mohr's integral (3).

$$\Delta_\eta \le \frac{q_b l^4 \mathrm{P}_1}{E\frac{h_0}{\lambda_w}\left(h_0^3\right)\left(\frac{1}{2}k_b + \frac{1}{6}\right)} \int_0^1 \frac{f_{M,z}^m\left(1 - \frac{z}{l}\right)}{f_{hz}^3} d\frac{z}{l} \quad (7)$$

The formulated problem is ultimately stated as a nonlinear programming problem: to determine the minimum normalized steel consumption for a cantilever I-beam structure with variable web height and flange width, subject to serviceability limit state constraints and the rational (optimal) distribution of steel along the cross-sectional height (5) as well as deflection constraints (7).

$$\frac{m_b}{\rho l^3} = \int_0^1 \frac{h_0^2}{\lambda_w l^2} f_{hz}(k_b + 1) d\frac{z}{l} \rightarrow min. \quad (8)$$

Condition (7), expressed in dimensionless form, has the following analytical representation.

$$\Delta_\eta \le \frac{q_b l^4 \mathrm{P}_1}{E\frac{h_0}{\lambda_w}\left(h_0^3\right)\left(\frac{1}{2}k_b + \frac{1}{6}\right)} \int_0^1 \frac{f_{M,z}^m\left(1 - \frac{z}{l}\right)}{f_{hz}^3} d\frac{z}{l}.$$

$$\frac{\Delta_\eta}{l}\frac{h_0}{\lambda_w l}\frac{El\left(\frac{1}{2}k_b + \frac{1}{6}\right)}{q_b} \le \int_0^1 \frac{f_{M,z}^m\left(1 - \frac{z}{l}\right)}{\left(h_0^3\right) f_{hz}^3 / l^3} d\frac{z}{l} \quad (9)$$

Euler's equation takes the following form.

$$F'_{h(z)}(\lambda_{m\eta}, A_z) - \frac{d}{dz} F'_{h'(z)}(\lambda_{m\eta}, A_z) + \ldots = 0 \tag{10}$$

The Lagrangian function from Equation (10) along with the deflection constraint from Equation (9 (non-rigidity condition, compatibility condition), is expressed in the following form (11)

$$F(\lambda_{m\eta}, A_z) = \frac{h_0 f_{hz}}{l} \frac{h_0}{\lambda_w l}(k_b + 1) + \lambda_{m\eta} \left[ \frac{f^m_{M,z}\left(1 - \frac{z}{l}\right)}{\left(h_0^3 f_{hz}^3\right)/l^3} \right].$$

$$\int_0^1 \frac{f^m_{M,z}\left(1 - \frac{z}{l}\right)}{\frac{h_0^3 f_{hz}^3}{l^3}} d\left(\frac{z}{l}\right) - \frac{\Delta_\eta}{l} \frac{h_0}{\lambda_w l} \frac{El}{q_b}\left(\frac{1}{2}k_b + \frac{1}{6}\right) = 0$$

For the formulated problem, taking into account the relationships between the geometric characteristics, Euler's equation takes the following form after differentiation

$$\frac{\partial\left[\frac{h_0 f_{hz}}{l} \frac{h_0}{\lambda_w l}(k_b + 1)\right]}{\partial\left(\frac{h_0 f_{hz}}{l}\right)} + \lambda_{m\eta} \frac{\partial\left(\lambda_{m\eta} \frac{f^m_{M,z}\left(1 - \frac{z}{l}\right)}{\frac{h_0^3 f_{hz}^3}{l^3}} d\left(\frac{z}{l}\right)\right)}{\partial\left(\frac{h_0 f_{hz}}{l}\right)} = 0.$$

$$\frac{h_0}{\lambda_w l}(k_b + 1) - \lambda_{m\eta} f^m_{M,z}\left(1 - \frac{z}{l}\right) \frac{3h_0^2 f_{hz}^2 / l^2}{\left(h_0^3 f_{hz}^3 / l^3\right)^2} = 0.$$

Thus, Euler's equation has been obtained in a generalized form for steel beams with variable cross-section.

Task 1. At this stage of the research, it is more rational to consider cantilever beams with variable cross-section

Determine the optimal topology of a cantilever steel I-beam (welded section) under a parabolic distribution of the bending moment along the length of the structure.

$$M_{x,z} = \left(1 - \frac{z}{l}\right)^2 \rightarrow f^m_{M,z} = \left(1 - \frac{z}{l}\right)^2.$$

The Euler's equation, taking into account the assumed law of bending moment variation along the beam's length, takes the following form.

$$\frac{h_0}{\lambda_w l}(k_b+1)\left(\frac{hf_{hz}}{l}\right)^4-3\lambda_{m\eta}\left(1-\frac{z}{l}\right)^3=0. \tag{11}$$

The pattern of variation in beam height as a function of the coefficient $\lambda_m$ is described by the following analytical equation.

$$\frac{hf_{hz}}{l}=\lambda_{m\eta}^{1/4}\left(\frac{3}{\frac{h_0}{\lambda_w l}(k_b+1)}\right)^{1/4}\left(1-\frac{z}{l}\right)^{3/4}. \tag{12}$$

$$\left(\frac{hf_{hz}}{l}\right)^3=\lambda_{m\eta}^{3/4}\left(\frac{3}{\frac{h_0}{\lambda_w l}(k_b+1)}\right)^{3/4}\left(1-\frac{z}{l}\right)^{9/4}.$$

The deflection constraint (compatibility equation, non-rigidity condition) can be written in the following form after substituting expression (12) into the integral of condition (9).

$$\frac{\Delta_\eta}{l}\frac{El}{q_b}\frac{h_0}{l\lambda_w}\left(\frac{1}{2}k_b+\frac{1}{6}\right)=\left[\frac{h_0}{3\lambda_w l}\frac{(k_b+1)}{\lambda_{m\eta}}\right]^{3/4}\int_0^1\left(1-\frac{z}{l}\right)^{3/4}d\left(\frac{z}{l}\right)$$

After integration, we obtain the formula for determining the value of the coefficient $\lambda_m$.

$$\lambda_{m\eta}=\frac{7(k_b+1)\left(\frac{\lambda_w l}{h_0}\right)^{1/3}\left(\frac{l}{\Delta_\eta}\frac{q_b}{El}\right)^{4/3}}{12\left(\frac{1}{2}k_b+\frac{1}{6}\right)^{4/3}}\left(1-\frac{z}{l}\right)^{7/3} \tag{13}$$

We substitute the coefficient коефіцієнта $\lambda_m$ in Euler’s equation (11) using formula (13).

$$\frac{h_0^4 f_{hz}^4}{l^4}=\frac{7\left(\frac{l}{\Delta_\eta}\frac{q_b}{El}\frac{\lambda_w l}{h_0}\right)^{4/3}}{4\left(\frac{1}{2}k_b+\frac{1}{6}\right)^{4/3}}\left(1-\frac{z}{l}\right)^{16/3}.$$

Ultimately, the relative height of the steel beam’s cross-section along its length, depending on the design conditions (compliance with deflection and strength constraints), follows a power-law relationship.

$$\frac{h_0 f_{hz}}{l} = \left(\frac{7}{4}\right)^{1/4} \left( \frac{\lambda_w l}{h_0 \left(\frac{1}{2}k_b + \frac{1}{6}\right)} \frac{l}{\Delta_\eta} \frac{q_b}{El} \right)^{1/3} \left(1 - \frac{z}{l}\right)^{4/3}. \quad (14)$$

Alternatively, by referring back to the web thickness of the maximum cross-section:

$1/t_w = \lambda_w / h_0$

$$\frac{h_0 f_{hz}}{l} = \left(\frac{7}{4}\right)^{1/4} \left( \frac{l}{t_w \left(\frac{1}{2}k_b + \frac{1}{6}\right)} \frac{l}{\Delta_\eta} \frac{q_b}{El} \right)^{1/3} \left(1 - \frac{z}{l}\right)^{4/3}. \quad (15)$$

Thus, the rational height of a beam with variable flange width and web height follows a power-law dependence (14,15). Interestingly, this power-law relationship is closely approximated by a linear function, which opens the possibility of approaching the optimal topology by adjusting the gradient of geometric variation of the flange and web — as demonstrated in [7,8] based on the criterion of steel consumption.

The derived analytical expressions (14),(15) make it possible, at the initial stage of variant-based design, to determine the rational height of the maximum cross-section.

Task 2. Determination of the rational height

of the support cross-section of a cantilever steel beam with variable flange width and web height.

To solve this problem, it is necessary to equate the algebraic formulas (15) and (1). In addition, the coordinate of the initial cross-section must be set as z=0.

$$\frac{h_0 f_{hz}}{l}\left(1 - \gamma_h \frac{z}{l}\right)^s = \left(\frac{7}{4}\right)^{1/4} \left( \frac{l}{t_w} \frac{l}{\Delta_\eta} \frac{q_b}{El} \frac{1}{\left(\frac{1}{2}k_b + \frac{1}{6}\right)} \right)^{1/3} \left(1 - \frac{z}{l}\right)^{4/3}.$$

Two important relationships can be derived from the last equation.

$$\left(1 - \gamma_h \frac{z}{l}\right)^s = \left(1 - \frac{z}{l}\right)^{4/3} \rightarrow s = 4/3 \quad (16)$$

$$\frac{h_0}{l} = \left(\frac{7}{4}\right)^{1/4} \sqrt[3]{\frac{l}{t_w \left(\frac{1}{2}k_b + \frac{1}{6}\right)} \frac{q_b}{El} \frac{l}{\Delta_\eta}} \quad (17)$$

The analytical formula (16) makes it possible to determine the degree of cross-sectional variation of the I-beam along the length of the structure at the very first stage of design.

Formula (17) allows for the determination of the rational height of the cross-section of an I-beam with variable web height and flange width at the maximum section, where the bending moment is applied.

## CONCLUSIONS AND PROSPECTS FOR FURTHER RESEARCH

The analytical relationships (14, 15) describing the rational topology of a steel cantilever I-beam with variable cross-section have been derived, based on deflection constraints and the optimality conditions for each cross-section in terms of steel consumption. An additional analytical formula has also been obtained for determining the rational height of the maximum cross-section of a tapered I-beam at the initial stage of structural design.

The general form of the analytical expressions (14,15) provides the opportunity, when transitioning to a piecewise-linear variation of the I-beam cross-section with variable web height and flange width, to determine the optimal structural topology in terms of steel consumption, while satisfying deflection constraints.

The obtained results can also be used in the analysis of the development of limited plastic deformations by incorporating material nonlinearity into the analytical expression for the variation of the cross-sectional moment of inertia.

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# РАЦІОНАЛЬНА ТОПОЛОГІЯ СТАЛЕВИХ КОНСОЛЬНИХ БАЛОК ЗІ ЗМІННОЮ ШИРИНОЮ ПОЛИЦЬ І ВИСОТОЮ СТІНКИ ПРИ ОБМЕЖЕННЯХ ПО ПРОГИНУ ТА МІЦНОСТІ

*Любомир ДЖАНОВ*

**Анотація.**У статті розв'язується задача вибору оптимальної топології сталевої консольної двотаврової балки зі змінною висотою стінки та шириною полиць за умови обмежень по прогину та за припущенням оптимального розподілу сталі в кожному поперечному перерізі відповідно до умов міцності. Задача розв'язується методом множників Лагранжа. В якості критерію оптимального проєктування приймається цільова функція мінімізації витрати сталі на конструкцію. Для кожного поперечного перерізу прийнято умову оптимального розподілу сталі між полицями та стінкою на основі критеріїв міцності. Можливі відхилення від оптимального співвідношення між площею полиць і площею стінки враховуються додатковим коефіцієнтом. Задача належить до класу задач нелінійного програмування. Умова міцності стінки вважається неактивною і забезпечується конструктивними заходами (ребрами жорсткості стінки). Позаплощинна стійкість балки (до бокового вигину) забезпечується відповідною системою горизонтальних зв'язків уздовж полиць.

Отримано аналітичну функцію, що описує зміну поперечного перерізу вздовж довжини консольної змінноперерізної двотаврової балки за умов оптимізації при рівномірно розподіленому навантаженні та заданому відносному проєктному прогині. Виведена аналітична функція для відносної оптимальної висоти двотавра вздовж довжини конструкції є степеневою та залежить від навантаження, обмежень по прогину і оптимального або раціонального розподілу сталі в кожному перерізі. Оптимальна висота двотаврової балки для опорного перерізу (в якому виникає максимальний згинальний момент) визначається згідно з умовами задачі.

Виявлені закономірності зміни оптимальної висоти балки дозволяють визначити раціональну топологію конструкції та врахувати можливість виникнення підвищених напружень у перерізах з меншою висотою. Підтверджено, що оптимальне конструктивне рішення залежить від закону розподілу навантаження. Отримані результати дозволяють визначити ступінь змінності висоти поперечного перерізу для оптимальної топології.

Виведені аналітичні формули для оптимальної висоти балки зі змінною шириною полиць та змінною висотою стінки дають змогу вже на першому етапі варіантного проєктування оцінити ефективність запропонованого конструктивного рішення.

**Ключові слова:** пружна сталева балка; консольна балка; оптимізація за витратою сталі при обмеженнях по прогину; оптимальні поперечні перерізи двотавра за умовами міцності; метод Лагранжа.

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